Homology of groups and third busy beaver function

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Abstract

In 2007, Nabutovsky and Weinberger provided a solution to a long-standing problem: to find naturally defined functions that grow faster than any function with Turing degree of unsolvability $\mathbf{0'}$. They considered the functions b_k such that, for a natural integer N, $b_k(N)$ is the rank of the kth homology group $H_k(G)$ of maximum finite rank, among the finitely presented groups G with presentation length $\leq N$. They proved that, for $k \geq 3$, function b_k grows as the third busy beaver function, and so grows faster than any function with degree of unsolvability $\mathbf{0''}$.

Can more be said about these functions b_k? We give some results on the function b_2, we study the challenge of computing $H_k(G)$ for a finitely presented group G, and we compute $b_k(N)$ for small values of N.

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1 Introduction

How can one write big natural numbers with few symbols? Ever since Archimedes, who was the first to tackle this problem, many authors have defined notations designed to fulfill the task, e.g., *Ackermann's function* [2, 15], *Knuth's arrow notation* [16, 5], *Conway's chained arrow notation* [8].

A step higher than these is *Rado's busy beaver function* [26, 20, 23], a noncomputable function that grows faster than any computable function. This function Σ is defined as follows: For any fixed natural number n, consider the Turing machines with a single tape, two symbols (0 and 1), and n states (plus a halting state). Each Turing machine from this finite set is launched on the empty tape (that is, the tape filled with 0s). Those that halt leave some symbols 1 on the tape when halting. Then $\Sigma(n)$ is the maximum number of 1 left on the tape by halting Turing machines with n states when they halt. Another noncomputable function S can be defined: $S(n)$ is the maximum number of computation steps made by Turing machines with n states that halt when they are launched on the empty tape. It is clear that the Turing machines used in these definitions are determined by a small number of conventions, so $\Sigma(n)$ and $S(n)$ have well defined values for $n = 1, 2, 3, \ldots$.

Then the *second busy beaver function* can be defined by Turing machines that use the busy beaver function as an oracle, and of course, the $n+1$st busy beaver function can be defined by Turing machines that use the nth busy beaver function as an oracle. The trouble is that there is no canonical way to define a Turing machine with oracle, so, for $n \geq 2$, the nth busy beaver function does not have an explicit definition, and we cannot give its values at $1, 2, 3, \ldots$.

However, in 2007, Nabutovsky and Weinberger [25] defined functions b_k that grow as the third busy beaver function when $k \geq 3$, and do have an explicit definition, as follows: For any fixed group G, an infinite sequence of abelian groups, denoted by $H_k(G)$, $k \geq 0$, can be defined. The group $H_k(G)$ is called the kth homology group of the group G. Like any abelian group, the group $H_k(G)$ has a rank, denoted by $b_k(G)$, which is the maximum integer r such that the group contains a subgroup isomorphic to $\mathbb{Z}^r$ (and the rank is infinite if such a maximum r does not exist). Now, consider the groups of length $\leq N$. A group is of length $\leq N$ if there is a finite presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of G, with generators $x_1, \ldots, x_n$ and relators $r_1, \ldots, r_m$, such that $n + \sum_{i=1}^{m} \text{length}(r_i) \leq N$. There is a finite number of groups of length $\leq N$. Then $b_k(N)$ is defined as the maximum rank $b_k(G)$

of a group $H_k(G)$, when G is taken among the groups of length $\leq N$ with a finite $b_k(G)$.

While the nth busy beaver function has no definite value at $1, 2, 3, \ldots$, because it has no formal definition, function $b_k(N)$ has an explicit definition. Until now, the biggest numbers that could be defined explicitly with few symbols were built from Rado's busy beaver functions. Function $b_k(N)$ enables us to define, explicitly and with few symbols, numbers bigger than those that have ever been written. Scott Aaronson asked for such numbers in his paper *Who can name the bigger number?* [1]. He suggested that nth busy beaver functions could provide such numbers and he asked for such functions that would have natural definitions.

For which N do the values $b_k(N)$ become really big numbers? To answer this question, a first step is to compute $b_k(N)$ for small values of N. In this article, we determine $b_k(N)$ for $0 \leq N \leq 9$. Is trying to compute the first values of a highly noncomputable function an idle occupation? It is a fact that the same task for Rado's busy beaver functions have turned out to be a rich and fruitful activity. Computer searches continue to need the development of nontrivial programs. The Turing machines involved in these works have been found to have interesting behaviors, which depend on well known open problems in number theory [21, 22]. We think that a study of functions $b_k(N)$ will lead to similar surprising discoveries..

In Section 2, we recall some notations in group theory. As we explain above and in the final section, our results may be of interest to the general mathematician, and not just for the expert in both homology theory and computation theory. So we give detailed preliminaries on these theories in Sections 3 and 4. In Section 5, we survey the known facts about computability of homology groups, and we present some new results. In Section 6, we present the main result of this paper: the computation of $b_k(N)$ when $0 \leq N \leq 9$. In Section 7, we give some lower bounds on functions $b_k(N)$. We conclude in Section 8 with some prospects for research.

2 Preliminaries: group theory

We will write groups both multiplicatively and additively. The unit is denoted by 1, and the trivial group $\{1\}$ is denoted by 0. Isomorphism of groups G_1 and G_2 is denoted by $G_1 \cong G_2$. The finite cyclic group of order n, $\mathbb{Z}/n\mathbb{Z}$, is denoted by $\mathbb{Z}_n$. The free product of groups G_1 and G_2 is denoted by

$G_1 * G_2$. The free product of group G with itself n times, $G * \cdots * G$, is denoted by G^{*n}. The direct product of groups G_1 and G_2 is denoted by $G_1 \times G_2$. The direct sum of groups G_1 and G_2 is denoted by $G_1 \oplus G_2$. Of course, $G_1 \times G_2 \cong G_1 \oplus G_2$, but we use both notations to make a difference between the studied groups and their homology groups. The direct product of group G with itself n times, $G \times \cdots \times G$, is denoted by G^n, as usual.

Let F be the free group on the set of generators $X = \{x_1, x_2, \ldots\}$. Let $X^{-1} = \{x_1^{-1}, x_2^{-1}, \ldots\}$, let $r_1, r_2, \ldots$ be words with letters in $X \cup X^{-1}$, and let R be the normal closure of $\{r_1, r_2, \ldots\}$ in F. Then the group $G = F/R$ is said to have a *presentation* with *generators* $\{x_1, x_2, \ldots\}$ and *relators* $\{r_1, r_2, \ldots\}$, and is denoted by $G = \langle x_1, x_2, \ldots | r_1, r_2, \ldots \rangle$. The group G is *finitely presented* if it has a finite presentation $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ with finite numbers of generators and relators. The free group with n generators is $F_n = \langle x_1, \ldots, x_n | \emptyset \rangle \cong \mathbb{Z}^{*n}$. The free group with infinite denumerable generators is denoted by F_∞ (see [14, 19] for more details).

The *length* of a finite presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ is $n + \sum_{i=1}^{m} \text{length}(r_i)$. A group is of length $\leq N$ if it has a finite presentation of length $\leq N$. A group is of length N if it is of length $\leq N$, but not of length $\leq N-1$. The length of G is denoted by $\text{length}(G)$.

A *commutator* in a group G is an element $[x, y] = x^{-1}y^{-1}xy$ with $x, y \in G$. If A, B are nonempty subsets of G, then $[A, B]$ is the subgroup generated by the set of commutators $[a, b]$, with $a \in A$, $b \in B$. The subgroup $[A, G]$ is normal in G. The subgroup $[G, G]$, also denoted by G', is the *derived group* of group G. When N is a normal subgroup of G, the quotient G/N is abelian if and only if $[G, G] \subseteq N$. So $G/[G, G]$ is the largest abelian factor group of G, called the *abelianization* of G, and denoted by G_{ab}. We have $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle_{ab} \cong \langle x_1, \ldots, x_n | r_1, \ldots, r_m, ([x_i, x_j])_{1 \leq i < j \leq n} \rangle$.

3 Preliminaries: homology of groups

3.1 Definition and examples

Let G be a group. We give below two definitions for the sequence $H_1(G)$, $H_2(G)$, …, of homology groups with coefficients in $\mathbb{Z}$. Unfortunately, none of them is simple. See [12, 17, 27, 31] for more on homology of groups.

First definition. For any group G, there exists a path-connected topological space BG, called *classifying space*, or *space of type* $(G, 1)$, such that

$\pi_1(BG) \cong G$ and $\pi_k(BG) \cong 0$ if $k \neq 1$ [29]. The space BG is unique up to homotopy equivalence, so the nth integral homology group $H_n(BG)$ does not depend on the choice of BG. Then the nth homology group of G with coefficients in $\mathbb{Z}$ is defined by $H_n(G) = H_n(BG)$.

Second definition. Let $\mathbb{Z}G$ be the group ring of G. By definition, $\mathbb{Z}G = \{\sum_{g\in G} n_g g : n_g \in \mathbb{Z}, n_g = 0 \text{ for all but a finite number of } g\}$, with the obvious addition and multiplication. The additive group $\mathbb{Z}$ is a $\mathbb{Z}G$-module for the trivial action: $gn = n$ if $g \in G$ and $n \in \mathbb{Z}$. A *free resolution* of $\mathbb{Z}$ is an exact sequence

$$\cdots \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} \cdots M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

where M_n are free $\mathbb{Z}G$-modules, and $\text{Ker } d_n = \text{Im } d_{n+1}$.

Tensoring a free $\mathbb{Z}G$-module M_n by $\otimes_{\mathbb{Z}G}\mathbb{Z}$ is nothing but killing the action of G (because $(gx) \otimes k = x \otimes (gk) = x \otimes k$ if $g \in G$, $x \in M_n$, $k \in \mathbb{Z}$), so we get a (not necessarily exact) sequence of abelian groups

$$\cdots \xrightarrow{\partial_{n+1}} M_n \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{\partial_n} \cdots M_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{\partial_1} M_0 \otimes_{\mathbb{Z}G} \mathbb{Z}$$

where $\partial_n(x \otimes k) = (d_n x) \otimes k$ if $x \in M_n$ and $k \in \mathbb{Z}$. Then the nth homology group of G is defined by $H_n(G) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}$.

Example 1. Let $G = \langle x | x^k \rangle \cong \mathbb{Z}_k$. Then $\mathbb{Z}G = \{\sum_{i=0}^{k-1} n_i x^i : n_i \in \mathbb{Z}\}$. Let $s = x - 1 \in \mathbb{Z}G$, $t = 1 + x + \cdots + x^{k-1} \in \mathbb{Z}G$. Then $st = x^k - 1 = 0$. The exact sequence of $\mathbb{Z}G$-modules is defined as follows. Let $M_n = \mathbb{Z}G$ for $n \geq 0$. The homomorphism $d_0 : \mathbb{Z}G \to \mathbb{Z}$ is defined by $d_0(\sum_{i=0}^{k-1} n_i x^i) = \sum_{i=0}^{k-1} n_i$. If $u \in \mathbb{Z}G$, let d_u be the multiplication by u. Then, let $d_{2n+1} = d_s$ for $n \geq 0$, and $d_{2n} = d_t$ for $n \geq 1$. It is easy to see that $\text{Im } d_s = \text{Ker } d_t = (\mathbb{Z}G)s$ and $\text{Im } d_t = \text{Ker } d_s = (\mathbb{Z}G)t$, so the following sequence is exact

$$\cdots \xrightarrow{d_t} \mathbb{Z}G \xrightarrow{d_s} \cdots \mathbb{Z}G \xrightarrow{d_t} \mathbb{Z}G \xrightarrow{d_s} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0.$$

Tensoring by $\otimes_{\mathbb{Z}G}\mathbb{Z}$, we get $M_n \otimes_{\mathbb{Z}G} \mathbb{Z} \cong \mathbb{Z}$, $\partial_{2n+1} = \partial_s = 0$, and $\partial_{2n} = \partial_t$ is multiplication by k. So $\text{Im } \partial_{2n+1} \cong 0$, $\text{Ker } \partial_{2n+1} \cong \mathbb{Z}$, $\text{Im } \partial_{2n} \cong k\mathbb{Z}$, $\text{Ker } \partial_{2n} \cong 0$, $H_{2n+1}(\mathbb{Z}_k) = \text{Ker } \partial_{2n+1}/\text{Im } \partial_{2n+2} \cong \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$, and $H_{2n}(\mathbb{Z}_k) = \text{Ker } \partial_{2n}/\text{Im } \partial_{2n+1} \cong 0$.

Example 2. Let $G = \langle x, y | xyx^{-1}y \rangle$. We define $d_0 : \mathbb{Z}G \to \mathbb{Z}$ by $d_0(\sum_g n_g g) = \sum_g n_g$, we define $d_1 : \mathbb{Z}G^2 \to \mathbb{Z}G$ by $d_1(1,0) = x-1$, $d_1(0,1) = y-1$, and we define $d_2 : \mathbb{Z}G \to \mathbb{Z}G^2$ by $d_2(1) = (yx^{-1}y - x^{-1}y, x^{-1}y + 1)$, using Fox derivative (for more details see for example [14]). Finally, we define $d_3 : 0 \to \mathbb{Z}G$. Then the following sequence is a free resolution of $\mathbb{Z}$

$$0 \xrightarrow{d_3} \mathbb{Z}G \xrightarrow{d_2} \mathbb{Z}G^2 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0.$$

Tensoring with $\mathbb{Z}$, we get

$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}$$

where $\partial_1(n, m) = 0$, $\partial_2(n) = (0, 2n)$. So $H_1(G) = \text{Ker } \partial_1/\text{Im } \partial_2 \cong \mathbb{Z} \oplus \mathbb{Z}_2$, and $H_n(G) \cong 0$ for $n \geq 2$.

The *rank* of an abelian group A is the greatest number $r \geq 0$ such that $\mathbb{Z}^r$ is a subgroup of A. It can also be defined as the maximum number of independent members of A, or as the dimension of the real vector space $A \otimes_{\mathbb{Z}} \mathbb{R}$. The rank is infinite if $\mathbb{Z}^r$ is a subgroup of A for all natural numbers r. The rank of the homology group $H_q(G)$ is denoted by $b_q(G)$ and is called the *qth Betti number* of G.

3.2 A toolkit to compute some homology groups

We will see in Section 5 that there is no algorithm to compute $H_n(G)$ from a finite presentation of group G, if $n \geq 2$. However, the following facts will enable us to compute homology groups for most of the cases we will meet.

(a) A zeroth homology group with coefficients in $\mathbb{Z}$ can be defined, with $H_0(G) \cong \mathbb{Z}$ for all groups G.

(b) $H_1(G) \cong G_{ab} = G/[G, G]$, the abelianization of G.

(c) Let $G = F/N$ with F a free group, and N a normal subgroup of F. Then *Hopf's formula* gives

$$H_2(G) \cong (N \cap [F, F])/[N, F].$$

(d) *Infinite cyclic group.*

$$H_1(\mathbb{Z}) \cong \mathbb{Z},$$

and, if $q \geq 1$,

$$H_q(\mathbb{Z}) \cong 0.$$

(e) *Finite cyclic groups.* If $q \geq 1$, then

$$H_{2q}(\mathbb{Z}_n) \cong 0,$$

and if $q \geq 0$, then

$$H_{2q+1}(\mathbb{Z}_n) \cong \mathbb{Z}_n$$

(see Example 1 above).

(f) *Free group with n generators.*

$$H_1(F_n) \cong \mathbb{Z}^n,$$

and, if $q \geq 2$,

$$H_q(F_n) \cong 0.$$

(g) *Free abelian group with n generators.* If n, $q \geq 1$, then

$$H_q(\mathbb{Z}^n) \cong \bigwedge^q \mathbb{Z}^n \cong \mathbb{Z}^{\binom{n}{q}}.$$

(h) *Free product of groups.* If $q \geq 1$, then

$$H_q(G_1 * G_2) \cong H_q(G_1) \oplus H_q(G_2).$$

(i) *Direct product of groups.* If $n \geq 1$, then

$$H_n(G_1 \times G_2) \cong \left(\bigoplus_{p+q=n} H_p(G_1) \otimes_{\mathbb{Z}} H_q(G_2) \right) \oplus \left(\bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(G_1), H_q(G_2)) \right),$$

with $p, q \geq 0$.

The following isomorphisms can help to compute this expression.

For any $\mathbb{Z}$-modules A, B, we have

$$A \otimes_{\mathbb{Z}} B \cong B \otimes_{\mathbb{Z}} A,$$

$$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C),$$
$$0 \otimes_{\mathbb{Z}} A \cong 0,$$
$$\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A,$$

$$A^p \otimes_{\mathbb{Z}} B^q \cong (A \otimes_{\mathbb{Z}} B)^{pq},$$

$$\mathbb{Z}_a \otimes_{\mathbb{Z}} \mathbb{Z}_b \cong \mathbb{Z}_{\gcd(a,b)} \quad \text{if } a, b \geq 2.$$

The same isomorphisms hold for $\mathrm{Tor}(A, B)$, except for the fourth one:

$$\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A),$$

$$\mathrm{Tor}(A, B \oplus C) \cong \mathrm{Tor}(A, B) \oplus \mathrm{Tor}(A, C),$$
$$\mathrm{Tor}(0, A) \cong 0,$$
$$\mathrm{Tor}(\mathbb{Z}, A) \cong 0,$$
$$\mathrm{Tor}(A^p, B^q) \cong \mathrm{Tor}(A, B)^{pq},$$
$$\mathrm{Tor}(\mathbb{Z}_a, \mathbb{Z}_b) \cong \mathbb{Z}_{\gcd(a,b)} \quad \text{if } a, b \geq 2.$$

For example, if $G_1 = \mathbb{Z}$, $G_2 = G$, then for all $p, q \geq 0$,

$$\mathrm{Tor}(H_p(\mathbb{Z}), H_q(G)) \cong 0,$$

and for all $p \geq 2$,

$$H_p(\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(G) \cong 0,$$

so

$$H_n(\mathbb{Z}\times G) \cong (H_0(\mathbb{Z})\otimes_{\mathbb{Z}} H_n(G))\oplus(H_1(\mathbb{Z})\otimes_{\mathbb{Z}} H_{n-1}(G)) \cong H_n(G)\oplus H_{n-1}(G).$$

From this we deduce that, for all $r \geq 2$,

$$H_1(\mathbb{Z} \times \mathbb{Z}_r) \cong \mathbb{Z} \oplus \mathbb{Z}_r,$$

and, if $n \geq 2$, then

$$H_n(\mathbb{Z} \times \mathbb{Z}_r) \cong \mathbb{Z}_r.$$

(j) *Finite groups.* If G is a finite group of order r, then, for all $q \geq 1$, $H_q(G)$ is a finite abelian group of exponent r (the exponent of an abelian group A is the least $n > 0$ with $nA = \{0\}$).

(k) *One-relator groups.* Let $G = \langle x_1, \ldots, x_n | r \rangle$ be a finitely presented one-relator group.

If r is not a proper power, then

$$H_2(G) \cong \begin{cases} \mathbb{Z} & \text{if } r \in [F_n, F_n] \\ 0 & \text{if } r \notin [F_n, F_n] \end{cases}$$

and

$$H_q(G) \cong 0 \quad \text{if } q \geq 3.$$

If $r = s^m$ with m maximum, $m \geq 2$, then

$$H_2(G) \cong \begin{cases} \mathbb{Z} & \text{if } s \in [F_n, F_n] \\ 0 & \text{if } s \notin [F_n, F_n] \end{cases}$$

$$H_{2q}(G) \cong 0 \quad \text{if } q \geq 2,$$

and

$$H_{2q+1}(G) \cong \mathbb{Z}_m \quad \text{if } q \geq 1.$$

4 Preliminaries: computability and busy beaver functions

The Turing machines we consider have a two way infinite tape, made of cells. Each cell contains a symbol. A machine has a tape head which moves on the tape, reading and writing symbols on the cells. A machine has a finite number of states and performs computations made of steps. At each step, according to the current state and the symbol read on the current cell, the machine writes on this cell, moves one cell left or right, and enters a new state. The computation stops when a special state H is reached.

Formally, a Turing machine is defined by the next move function, which is a mapping

$$\delta : Q \times S \longrightarrow S \times \{\text{Left, Right}\} \times (Q \cup \{H\}),$$

where $Q = \{A, B, \ldots\}$ is the finite set of states, $S = \{0, 1, \ldots\}$ is the finite set of symbols, {Left, Right} is the set of directions, and H is the final state. By convention, $H \notin Q$. If $\delta(p, a) = (b, X, r)$, then it means that, if the Turing machine is in state p, reading symbol a on the current cell, then it writes symbol b on the cell instead of a, it moves to the cell next to the current cell, in the direction $X \in \{\text{Left, Right}\}$, and it enters state r. Initially, the state is the initial state A, the cells contains a word of finite length called the input, and the tape head scans the first letter of the input. If the final state H is reached, then the machine stops. Then the word written on the tape is the output. If input and output are codes for natural numbers, then the Turing machine computes a function on natural numbers. The Church-Turing Thesis states that Turing machines provide a universal model of computation. That is, a function on natural numbers is *computable* (or *recursive*) if and only if it is computable by a Turing machine. A set of natural numbers is *computably enumerable* (or *recursively enumerable*) if it is the range of a computable function.

Now, consider Turing machines with two symbols, so $S = \{0, 1\}$. There are $(4(n+1))^{2n}$ Turing machines with two symbols and n states. Each of them can be launched on a blank tape, that is a tape filled with the blank symbol 0. Then each machine can either reach state H and stop, or else never stop. The machines that stop are called *busy beavers*. If M is a busy beaver, we denote by $s(M)$ the number of steps taken by M to stop, and by $\sigma(M)$ the number of symbols 1 left on the tape by M when it stops. The busy beavers with n states compete in two competitions: to take the maximum number of steps to stop, and to leave the maximum number of symbols 1 on the tape when stopping. So two busy beaver functions can be defined:

$$S(n) = \max\{s(M) : M \text{ is a busy beaver with } n \text{ states}\},$$

$$\Sigma(n) = \max\{\sigma(M) : M \text{ is a busy beaver with } n \text{ states}\}.$$

It is known that functions S and Σ are not computable, and grow faster than any computable function. That is, for any computable function f, there is a natural number n_0 such that, for all $n \geq n_0$, we have $S(n) > f(n)$. The values of $S(n)$ and $\Sigma(n)$ are known for $n = 2, 3, 4$ [26, 18, 7], and are still the subject of active research for $n = 5, 6$ [20, 23].

An oracle Turing machine has an additional tape, the oracle tape, that contains some information which is called the oracle. We will not give a formal definition of an oracle Turing machine, because there is no agreement on such a definition (see [28] for an example of definition). If the oracle is computable, then it is useless, because it could have been computed directly by the Turing machine. So the oracle is useful when it is not computable. For example, it can be the *halting problem* for Turing machines: given (M, x), where M is a (code for a) Turing machine, and x an input, the oracle says whether or not M stops on input x. Or it can be the busy beaver function S. It can be proved that a Turing machine with oracle the halting problem can compute S, and that a Turing machine with oracle S can solve the halting problem. So both belong to the same degree of unsolvability.

Formally, we write $A \leq_T B$ if A can be computed by a Turing machine with oracle B. We write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. The *degree of unsolvability* of A is $\deg(A) = \{B : B \equiv_T A\}$. The halting problem is denoted by K_0, and the halting problem for a Turing machine with oracle A is denoted by K_0^A, or by A', and is called the *jump* of A. Note that $K_0 = \emptyset'$. The nth jump of A, denoted by $A^{(n)}$, is defined by iteration: $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$. We denote $\mathbf{0}^{(n)} = \deg(\emptyset^{(n)})$. So $\mathbf{0}$ is the degree of computable sets and functions, and $\mathbf{0}'$ is the degree of K_0 and also of function S. The following lemma will be useful (see [28]).

Lemma 4.1 *A function $f(n)$ is computable with oracle A' if and only if there is a function $g(n,k)$ computable with oracle A, such that, for all n, $f(n) = \lim_{k\to\infty} g(n,k)$.*

Following Nabutovsky and Weinberger [25], we use the following definitions. A *Turing machine of order* 1 is a Turing machine without oracle, and, for $k \geq 2$, a *Turing machine of order k* is a Turing machine that has as oracle the halting problem for the Turing machines of order $k-1$. The *kth busy beaver function $B_k(n)$* is the maximum number of steps taken by a Turing machine of order k with n states and two symbols that stops when it is launched on a blank tape. So $B_1(n)$ is the usual busy beaver function $S(n)$ as defined above. Note that these definitions are informal ones, since we have not given a formal definition for an oracle Turing machine. In particular, $B_k(n)$ has no definite value for $k, n \geq 2$. Nonetheless, the results below will be true for any reasonable choice of a formal definition for an oracle Turing machine. Let us recall that a function f *grows faster* than a function g if there exists a natural number n_0 such that, for all $n \geq n_0$, $f(n) > g(n)$.

Proposition 4.2 *(i) A function is computable by a Turing machine of order k if and only if it is computable with oracle $\emptyset^{(k-1)}$.*

(ii) The function B_k is computable with oracle $\emptyset^{(k)}$ and grow faster than any function computable with oracle $\emptyset^{(k-1)}$.

(iii) *The halting problem for Turing machines of order k and the computation of B_k belong to the same degree of unsolvability $\mathbf{0}^{(k)}$.*

5 Computability of homology groups

The fact that homology groups are generally not computable may seem surprising, because the definition of homology groups by free resolution and tensoring seems to give an algorithm, and one can ask where the noncomputability comes in. The answer is that it is the passage from a finite presentation of a group to its multiplication table that is not computable. There is no algorithm to solve the *word problem*: Given a finite presentation of a group and a word made from the generators, does this word represent the unit element of the group?

5.1 Computability of $H_1(G)$ and $H_2(G)$

We first consider finitely presented groups. The computation of $H_1(G)$ from a finite presentation of a group G is well known (see e.g. [24]), and can be summed up as follows.

Theorem 5.1 *Let G be finitely presented group. Then*

(i) *$H_1(G) \cong G_{ab}$ is a finitely generated abelian group with a normal form which is computable from the finite presentation of G.*

(ii) *$b_1(G)$ is computable from the finite presentation of G.*

(iii) *For all $N \geq 0$, we have $b_1(N) = N$.*

Proof. Let $G = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ be a finitely presented group, given by its finite presentation. Then $H_1(G) \cong G_{ab}$ is a finitely generated abelian group. So it has the normal form $H_1(G) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_l} \oplus \mathbb{Z}^k$, where k,$l \geq 0$, $d_i \geq 2$ and d_i divides d_{i+1} for all i. Moreover, k, l and the d_i can be effectively computed from the finite presentation of G. In particular, $b_1(G) = k$, so the function $b_1(G)$ is computable from a finite presentation of G. Then $b_1(N)$ is computable by computing $b_1(G)$ for all groups G of lengths $\leq N$ and taking the maximum result.

If $\langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ is a finite presentation of group G with minimal length, then G_{ab} has at most n generators, so $b_1(G) \leq n \leq \text{length}(G)$. On the other hand, for $G = F_n = \langle x_1, \dots, x_n | \emptyset \rangle$, we have $H_1(G) \cong \mathbb{Z}^n$, so $b_1(G) = n = \text{length}(G)$. Thus, the maximum value of $b_1(G)$ for a group G of length $\leq N$ is N, achieved for $G = F_N$. $\square$

Let us consider $H_2(G)$ for a finitely presented group G. The situation becomes more complicated, as shown by the following theorem, where statements (iv) to (vi) are new results.

Theorem 5.2 *Let G be a finitely presented group. Then*

(i) $H_2(G)$ is a finitely generated abelian group, with a number of generators less than or equal to the number of relators of G.

(ii) A normal form for $H_2(G)$ can be computed with oracle $\emptyset'$.

(iii) There is no algorithm to decide, given a finite presentation of G, whether $H_2(G) \cong 0$.

(iv) There is no algorithm to decide, given a finite presentation of G, whether $b_2(G) = 0$.

(v) The function which maps a finite presentation of G to $b_2(G)$ is not computable, but is computable with oracle $\emptyset'$.

(vi) The function $N \mapsto b_2(N)$ is computable with oracle $\emptyset'$.

Proof. (i) We recall the proof of this well known result (see e.g. [24]). Let $G = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$. If N is the normal subgroup of the free group F_n on n generators, generated by the relators $r_1, \dots, r_m$, we have $G \cong F_n/N$, and Hopf's formula gives $H_2(G) \cong (N \cap [F_n, F_n])/[N, F_n]$. The group $N/[N, F_n]$ is an abelian group, generated by the set of cosets $\{r_i[N, F_n] : 1 \leq i \leq m\}$, so it is a finitely generated abelian group, with at most m generators. The group $(N \cap [F_n, F_n])/[N, F_n]$ is a subgroup of $N/[N, F_n]$, so it is a finitely generated abelian group with at most m generators.

(ii) Gordon [10] attributed to Casson the observation that a set of relators for $H_2(G)$ can be computably enumerated. From a finite presentation $\langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ of G, we can compute generators $y_1, \dots, y_m$ and relators $(s_i)_{i \geq 1}$ for $H_2(G)$, where the sequence $(s_i)_{i \geq 1}$ is computably enumerable. For any k, we can compute a normal form for the finitely generated abelian group $\langle y_1, \dots, y_m | s_1, \dots, s_k \rangle_{ab}$. So the functions h_k that compute such a normal form from a finite presentation of G are computable. By Lemma 4.1, the function $h = \lim_{k\to\infty} h_k$ is computable with oracle $\emptyset'$, and gives a normal form for $H_2(G)$ from a finite presentation of G.

(iii) This is the main result in Gordon's article [10].

(iv) This new result can be easily obtained from Gordon's methods, as follows. We resume the proof of Theorem 4 in Gordon's article [10]. From a finite presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of a group G with an unsolvable word problem, and a word w with letters in $\{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$, this proof defines a finitely presented group H_w such that

$$w = 1 \text{ in } G \Longrightarrow H_2(H_w) = 0,$$

$$w \neq 1 \text{ in } G \Longrightarrow \text{ there is a homomorphism that maps } H_2(H_w) \text{ onto } \mathbb{Z}.$$

So $H_2(H_w) = 0$ iff $w = 1$ in G, and it is undecidable whether $H_2(H_w) = 0$, thus we get the result stated above in (iii). But we have also

$$w = 1 \text{ in } G \Longrightarrow b_2(H_w) = 0,$$

$$w \neq 1 \text{ in } G \Longrightarrow b_2(H_w) \geq 1.$$

So $b_2(H_w) = 0$ iff $w = 1$ in G, and the question of whether $b_2(H_w) = 0$ is undecidable.

(v) This is clear from statements (ii) and (iv).

(vi) This comes from statement (v). With oracle $\emptyset'$, $b_2(G)$ can be computed for all groups in the finite set of groups of length $\leq N$. Then taking the maximum value gives $b_2(N)$. $\square$

It is not known whether $b_2(N)$ is a computable function of N. Note that $b_2(N) \leq N$ for all $N \geq 0$, since the number of generators of $H_2(G)$ is no larger than the number of relators of G. We conjecture that $b_2(N) \leq \left\lfloor \frac{N}{4} \right\rfloor$.

Consider now the groups that are finitely generated and have a computably enumerable presentation. That is, the set of relators is a computably enumerable set of words. By a theorem of Higman [13], these groups are exactly the finitely generated subgroups of finitely presented groups. Therefore, there are two ways to present such groups by a finite description. First, group G can be given by a finite program that enumerate the set of relators of G. Second, group G can be given by a finite presentation of a group K and a finite list of words from K that generate G as a subgroup of K. For such a group G, Baumslag, Dyer and Miller [3] proved that $H_q(G)$ has a computably enumerable presentation for all $q \geq 1$, but Bogley and Harlander [6] proved that there is no algorithm to decide whether $H_1(G) \cong 0$ or whether $H_2(G) \cong 0$.

5.2 Computability of $H_q(G)$ for $q \geq 3$

For $q \geq 3$, if G is finitely presented, then $H_q(G)$ is an abelian group with a computably enumerable presentation, and, moreover, for any abelian group A with a computably enumerable presentation, and any $q \geq 3$, there is a finitely presented group G such that $H_q(G) \cong A$ [3].

Let us say that function f *grows as* function g if there are computable functions φ and ψ, and an integer n_0 such that, for all $n \geq n_0$, $f(\varphi(n)) > g(n)$ and $g(\psi(n)) > f(n)$. Nabutovsky and Weinberger [25] proved the following theorem.

Theorem 5.3 *For $q \geq 3$, the function $b_q(N)$ grows as the third busy beaver function B_3.*

So we have the following result.

Corollary 5.4 *Let $q \geq 3$. Then*

(i) The function $N \mapsto b_q(N)$ is not computable with oracle $\emptyset''$.

(ii) The function which maps a finite presentation of the group G to $b_q(G)$ is not computable with oracle $\emptyset''$.

Proof. (i) Function B_3 grows faster than any function computable with oracle $\emptyset''$, and so does $b_q(N)$.

(ii) Consider the function that maps a finite presentation of G to $b_q(G)$. If it was computable with oracle $\emptyset''$, then so should be $b_q(N)$, by taking the maximum $b_q(G)$ among the groups of length $\leq N$. □

6 Values of $b_q(N)$ for $0 \leq N \leq 9$

In this section we prove the following theorem.

Theorem 6.1 *(i)* $b_2(N) = \begin{cases} 0 & \text{if } 0 \leq N \leq 5 \\ 1 & \text{if } 6 \leq N \leq 9 \end{cases}$

(ii) If $0 \leq N \leq 9$ and $q \geq 3$, then $b_q(N) = 0$.

Proof. The theorem is proved by the following procedure.

1. We enumerate the presentations $\langle x_1,\dots,x_n|r_1,\dots,r_m\rangle$ of length N.

2. For each one, we find if the corresponding group has length $< N$, or is isomorphic to an already found group of length N. Then we get the list of groups of length N.

3. We compute the homology groups $H_q(G)$, $q \geq 1$, for each group G from the list of groups of length N.

4. We compute the ranks $b_q(G)$ of these homology groups.

5. Then $b_q(N)$ is the maximum of these ranks.

This procedure is tedious but straightforward. We give some details for each stage.

Stage 1. To avoid some pitfalls in the enumeration of presentations, normal forms for relators must be used with precaution. For example, group $\mathbb{Z}_9$ has the obvious presentation $\langle x|x^9\rangle$ of length 10. But it has also the shorter one $\langle x,y|x^3y,y^3\rangle$ of length 9.

For future studies, we give the relators involving two generators x and y, of length ≤ 6, up to the following transformations: circular permutations, taking inverse, exchanges of x and y, of x and x^{-1} and of y and y^{-1}.

- Length 3: x^2y.
- Length 4: x^3y, x^2y^2, $(xy)^2$, $xyx^{-1}y$, $xyx^{-1}y^{-1}$.
- Length 5: x^4y, x^3y^2, x^2yxy, x^2yxy^{-1}, $x^2yx^{-1}y$, $x^2yx^{-1}y^{-1}$.
- Length 6:

x^5y	x^3yxy^{-1}	$x^2yx^2y^{-1}$	$x^2yx^{-2}y^{-1}$	$xyxyx^{-1}y^{-1}$
x^4y^2	$x^3yx^{-1}y$	x^2yxy^2	$x^2yx^{-1}y^{-2}$	$xyxy^{-1}x^{-1}y^{-1}$
x^3y^3	$x^3yx^{-1}y^{-1}$	x^2yxy^{-2}	$(xy)^3$	$xyx^{-1}yxy^{-1}$
x^3yxy	$(x^2y)^2$	$x^2yx^{-2}y$	$xyxyxy^{-1}$	

Stage 2. Most groups we find can be described using group $\mathbb{Z}$, cyclic groups $\mathbb{Z}_n$ and free products or direct products, so it is easy to detect isomorphic groups. As an example, we give the details for the groups of length 6 to 8 with a presentation $\langle x,y|r\rangle$ where r is a relator of length 4 to 6.

Groups $\langle x, y | r \rangle$ with r of length 4.

Six such relators have to be considered: x^4, x^2y^2, $xyxy$, $xyxy^{-1}$, $xyx^{-1}y$ and $xyx^{-1}y^{-1}$. But

- $\langle x, y | xyxy^{-1} \rangle \cong \langle y, x | yxy^{-1}x \rangle$,
- $\langle x, y | x^2y^2 \rangle \cong \langle u, v | uvu^{-1}v \rangle$ with $u = y^{-1}$ and $v = xy$,
- $\langle x, y | xyxy \rangle \cong \langle x, u | u^2 \rangle$ of length 4, with $u = xy$.

So three groups are left:

- $\langle x, y | x^4 \rangle \cong \mathbb{Z} * \mathbb{Z}_4$,
- $\langle x, y | xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2$,
- $\langle x, y | xyx^{-1}y \rangle$.

The last group is the semidirect product of normal subgroup $N = \{y^n : n \in \mathbb{Z}\}$ with subgroup $M = \{x^n : n \in \mathbb{Z}\}$. Both these subgroups are isomorphic to $\mathbb{Z}$, so we denote the group by $\mathbb{Z} \ltimes \mathbb{Z}$. The three groups $\mathbb{Z} * \mathbb{Z}_4$, $\mathbb{Z}^2$ and $\mathbb{Z} \ltimes \mathbb{Z}$ were not found in the study of groups of length ≤ 5, so they have length 6.

Groups $\langle x, y | r \rangle$ with r of length 5.

Seven relators have to be considered: x^5, x^4y, x^3y^2, x^2yxy, x^2yxy^{-1}, $x^2yx^{-1}y$, $x^2yx^{-1}y^{-1}$. But

- $\langle x, y | x^4y \rangle \cong \langle x | x^4 \rangle$ of length 5,
- $\langle x, y | x^3y^2 \rangle \cong \langle x, u | x^2ux^{-1}u \rangle$ with $u = xy$,
- $\langle x, y | x^2yxy \rangle \cong \langle x, u | xu^2 \rangle$ of length ≤ 5, with $u = xy$.

So four groups are left:

- $\langle x, y | x^5 \rangle \cong \mathbb{Z} * \mathbb{Z}_5$,
- $\langle x, y | x^2yxy^{-1} \rangle = K_1$,
- $\langle x, y | x^2yx^{-1}y \rangle = K_2$,
- $\langle x, y | x^2yx^{-1}y^{-1} \rangle \cong$ BS(1,2).

Group K_2 and the Baumslag-Solitar group BS(1,2) have isomorphic abelianizations. They are not isomorphic because they have derived groups that are not isomorphic: $K_2' \cong \mathbb{Z}^{*2}$ and $\text{BS}(1,2)' \cong \mathbb{Z}\left[\frac{1}{2}\right]$. The derived groups are determined by the Reidemeister-Schreier method, described for example in [14, 19].

Groups $\langle x, y | r \rangle$ with r of length 6.

The study led to the following nine groups.

- $\langle x, y | x^6 \rangle \cong \mathbb{Z} * \mathbb{Z}_6$,
- $\langle x, y | x^2 y x y^{-2} \rangle \cong K_3$,
- $\langle x, y | x^2 y x^{-1} y^{-2} \rangle \cong K_4$,
- $\langle x, y | x^2 y x^{-2} y^{-1} \rangle \cong \text{BS}(2,2)$,
- $\langle x, y | x^3 y x^{-1} y \rangle \cong K_5$,
- $\langle x, y | x^3 y x^{-1} y^{-1} \rangle \cong \text{BS}(1,3)$,
- $\langle x, y | x^3 y x y^{-1} \rangle \cong K_6$,
- $\langle x, y | x^2 y x^2 y^{-1} \rangle \cong K_7$,
- $\langle x, y | x^3 y^3 \rangle \cong K_8$.

Moreover, $K_3 \not\cong K_4$ because $K_3' \cong \mathbb{Z}^{*3}$, $K_4' \cong \mathbb{Z}^{*2}$, $K_5 \not\cong \text{BS}(1,3)$ because $K_5' \cong \mathbb{Z}^{*3}$, $\text{BS}(1,3)' \cong \mathbb{Z}\left[\frac{1}{3}\right]$, and $K_6 \not\cong K_7$ because $K_6' \cong \mathbb{Z}\left[\frac{1}{3}\right]$, $K_7' \cong \mathbb{Z} \times F_\infty$.

Stage 3. The computation of homology groups is easy using the tools from Section 3.2, with two exceptions: $\langle x, y | x^2 y x^{-1} y, y^2 \rangle \cong S_3$ (the symmetric group with 6 elements) and $\langle x, y | x y x^{-1} y, y^3 \rangle \cong \mathbb{Z}_3 \ltimes \mathbb{Z}$. It is known that, for all $n \geq 1$, $H_{2n}(S_3) \cong 0$, and, for all $n \geq 0$, $H_{4n+1}(S_3) \cong \mathbb{Z}_2$, $H_{4n+3}(S_3) \cong \mathbb{Z}_6$. The homology groups of $\mathbb{Z}_3 \ltimes \mathbb{Z}$ is given by the proposition below. We have $H_1(\mathbb{Z}_3 \ltimes \mathbb{Z}) \cong \mathbb{Z}$ and, if $q \geq 2$, $H_q(\mathbb{Z}_3 \ltimes \mathbb{Z}) \cong 0$ if $q \equiv 1, 2 \pmod 4$, $H_q(\mathbb{Z}_3 \ltimes \mathbb{Z}) \cong \mathbb{Z}_3$ if $q \equiv 0, 3 \pmod 4$.

Stages 4 and 5. The ranks of the homology groups and the maximum rank can be easily computed.

length	generators	relators	groups	H_1	H_{2n} $n \geq 1$	H_{2n+1} $n \geq 1$
0	$\emptyset$	$\emptyset$	0	0	0	0
1	x	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
2	x, y	$\emptyset$	$\mathbb{Z}^{*2}$	$\mathbb{Z}^2$	0	0
3	x, y, z	$\emptyset$	$\mathbb{Z}^{*3}$	$\mathbb{Z}^3$	0	0
	x, y	x^2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
4	x, y, z, t	$\emptyset$	$\mathbb{Z}^{*4}$	$\mathbb{Z}^4$	0	0
	x, y	x^2	$\mathbb{Z} * \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$
	x	x^3	$\mathbb{Z}_3$	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
5	x, y, z, t, u	$\emptyset$	$\mathbb{Z}^{*5}$	$\mathbb{Z}^5$	0	0
	x, y, z	x^2	$\mathbb{Z}^{*2} * \mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$
	x, y	x^3	$\mathbb{Z} * \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3$
	x	x^4	$\mathbb{Z}_4$	$\mathbb{Z}_4$	0	$\mathbb{Z}_4$

Table 1: The 12 groups of length ≤ 5

generators	relators	groups	H_1	H_2	H_{2n+1} $n \geq 1$	H_{2n} $n \geq 2$
x, y, z, t, u, v	$\emptyset$	$\mathbb{Z}^{*6}$	$\mathbb{Z}^6$	0	0	0
x, y, z, t	x^2	$\mathbb{Z}^{*3} * \mathbb{Z}_2$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
x, y, z	x^3	$\mathbb{Z}^{*2} * \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
x, y	x^4	$\mathbb{Z} * \mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_4$	0
	$xyx^{-1}y$	$\mathbb{Z} \ltimes \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0	0
	$xyx^{-1}y^{-1}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0
	x^2, y^2	$\mathbb{Z}_2^{*2}$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0
x	x^5	$\mathbb{Z}_5$	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	0

Table 2: The 8 groups of length 6

generators	relators	groups	H_1	H_2	H_{2n+1} $n \geq 1$	H_{2n} $n \geq 2$
x, y, z, t, u, v, w	$\emptyset$	$\mathbb{Z}^{*7}$	$\mathbb{Z}^7$	0	0	0
x, y, z, t, u	x^2	$\mathbb{Z}^{*4} * \mathbb{Z}_2$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
x, y, z, t	x^3	$\mathbb{Z}^{*3} * \mathbb{Z}_3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
x, y, z	x^4	$\mathbb{Z}^{*2} * \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_4$	0
	$xyx^{-1}y$	$\mathbb{Z} * (\mathbb{Z} \ltimes \mathbb{Z})$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	0	0	0
	$xyx^{-1}y^{-1}$	$\mathbb{Z} * \mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}$	0	0
	x^2, y^2	$\mathbb{Z} * \mathbb{Z}_2^{*2}$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0
x, y	x^5	$\mathbb{Z} * \mathbb{Z}_5$	$\mathbb{Z} \oplus \mathbb{Z}_5$	0	$\mathbb{Z}_5$	0
	x^2yxy^{-1}	K_1	$\mathbb{Z} \oplus \mathbb{Z}_3$	0	0	0
	$x^2yx^{-1}y$	K_2	$\mathbb{Z}$	0	0	0
	$x^2yx^{-1}y^{-1}$	BS(1,2)	$\mathbb{Z}$	0	0	0
	x^3, y^2	$\mathbb{Z}_2 * \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	0
x	x^6	$\mathbb{Z}_6$	$\mathbb{Z}_6$	0	$\mathbb{Z}_6$	0

Table 3: The 13 groups of length 7

The results are given in Tables 1 to 6. We give all groups of length ≤ 8 in Tables 1 to 4. In Tables 5 and 6, we give all groups of length 9, except groups with one relator of length 6 or 7 which is not a proper power of a relator. We know that such one-relator groups have homology groups $H_2(G) \cong 0$ or $\mathbb{Z}$ and $H_q(G) \cong 0$ if $q \geq 3$. $\square$

The homology groups of group $\langle x, y | xyx^{-1}y, y^3 \rangle$ of length 9 cannot be computed from the tools of Section 3.2. The following proposition gives the results, and has an independent interest.

Proposition 6.2 *Let* $G = \langle x, y | xyx^{-1}y, y^n \rangle$, $n \geq 2$. *Then*

$$H_1(G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

and, for all $q \geq 2$,

(i) if $q \equiv 0, 3 \pmod 4$, $H_q(G) \cong \mathbb{Z}_n$,

generators	relators	groups	H_1	H_2	H_{2n+1} $n \geq 1$	H_{2n} $n \geq 2$
$x, y, z, t,$ u, v, w, s	$\emptyset$	$\mathbb{Z}^{*8}$	$\mathbb{Z}^8$	0	0	0
$x, y, z, t,$ u, v	x^2	$\mathbb{Z}^{*5} * \mathbb{Z}_2$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
x, y, z, t, u	x^3	$\mathbb{Z}^{*4} * \mathbb{Z}_3$	$\mathbb{Z}^4 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
x, y, z, t	x^4	$\mathbb{Z}^{*3} * \mathbb{Z}_4$	$\mathbb{Z}^3 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_4$	0
	$xyx^{-1}y$	$\mathbb{Z}^{*2} * (\mathbb{Z} \ltimes \mathbb{Z})$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	0	0	0
	$xyx^{-1}y^{-1}$	$\mathbb{Z}^{*2} * \mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}$	0	0
	x^2, y^2	$\mathbb{Z}^{*2} * \mathbb{Z}_2^{*2}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0
x, y, z	x^5	$\mathbb{Z}^{*2} * \mathbb{Z}_5$	$\mathbb{Z}^2 \oplus \mathbb{Z}_5$	0	$\mathbb{Z}_5$	0
	x^2yxy^{-1}	$\mathbb{Z} * K_1$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	0	0	0
	$x^2yx^{-1}y$	$\mathbb{Z} * K_2$	$\mathbb{Z}^2$	0	0	0
	$x^2yx^{-1}y^{-1}$	$\mathbb{Z} *$ BS(1,2)	$\mathbb{Z}^2$	0	0	0
	x^3, y^2	$\mathbb{Z} * \mathbb{Z}_2 * \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	0
x, y	x^6	$\mathbb{Z} * \mathbb{Z}_6$	$\mathbb{Z} \oplus \mathbb{Z}_6$	0	$\mathbb{Z}_6$	0
	x^2yxy^{-2}	K_3	$\mathbb{Z}$	0	0	0
	$x^2yx^{-1}y^{-2}$	K_4	$\mathbb{Z}$	0	0	0
	$x^2yx^{-2}y^{-1}$	BS(2,2)	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0
	$x^3yx^{-1}y$	K_5	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0	0
	$x^3yx^{-1}y^{-1}$	BS(1,3)	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0	0
	x^3yxy^{-1}	K_6	$\mathbb{Z} \oplus \mathbb{Z}_4$	0	0	0
	$x^2yx^2y^{-1}$	K_7	$\mathbb{Z} \oplus \mathbb{Z}_4$	0	0	0
	x^3y^3	K_8	$\mathbb{Z} \oplus \mathbb{Z}_3$	0	0	0
	x^4, y^2	$\mathbb{Z}_2 * \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	0
	$xyx^{-1}y^{-1}, x^2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	x^3, y^3	$\mathbb{Z}_3^{*2}$	$\mathbb{Z}_3^2$	0	$\mathbb{Z}_3^2$	0
x	x^7	$\mathbb{Z}_7$	$\mathbb{Z}_7$	0	$\mathbb{Z}_7$	0

Table 4: The 25 groups of length 8

generators	relators	groups	H_1	H_2	H_{2n+1} $n \geq 1$	H_{2n} $n \geq 2$
$x, y, z, t, u, v, w, s, q$	$\emptyset$	$\mathbb{Z}^{*9}$	$\mathbb{Z}^9$	0	0	0
x, y, z, t, u, v, w	x^2	$\mathbb{Z}^{*6} * \mathbb{Z}_2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
x, y, z, t, u, v	x^3	$\mathbb{Z}^{*5} * \mathbb{Z}_3$	$\mathbb{Z}^5 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
x, y, z, t, u	x^4	$\mathbb{Z}^{*4} * \mathbb{Z}_4$	$\mathbb{Z}^4 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_4$	0
	$xyx^{-1}y$	$\mathbb{Z}^{*3} * (\mathbb{Z} \ltimes \mathbb{Z})$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	0	0	0
	$xyx^{-1}y^{-1}$	$\mathbb{Z}^{*3} * \mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}$	0	0
	x^2, y^2	$\mathbb{Z}^{*3} * \mathbb{Z}_2^{*2}$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0
x, y, z, t	x^5	$\mathbb{Z}^{*3} * \mathbb{Z}_5$	$\mathbb{Z}^3 \oplus \mathbb{Z}_5$	0	$\mathbb{Z}_5$	0
	x^2yxy^{-1}	$\mathbb{Z}^{*2} * K_1$	$\mathbb{Z}^3 \oplus \mathbb{Z}_3$	0	0	0
	$x^2yx^{-1}y$	$\mathbb{Z}^{*2} * K_2$	$\mathbb{Z}^3$	0	0	0
	$x^2yx^{-1}y^{-1}$	$\mathbb{Z}^{*2} *$ BS(1,2)	$\mathbb{Z}^3$	0	0	0
	x^3, y^2	$\mathbb{Z}^{*2} * \mathbb{Z}_2 * \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	0

Table 5: The groups of length 9 with more than 3 generators

gen.	relators	groups	H_1	H_2	H_{2n+1} $n \geq 1$	H_{2n} $n \geq 2$
x, y, z	x^6	$\mathbb{Z}^{*2} * \mathbb{Z}_6$	$\mathbb{Z}^2 \oplus \mathbb{Z}_6$	0	$\mathbb{Z}_6$	0
	one relator, length 6, not proper power			0 or $\mathbb{Z}$	0	0
	x^4, y^2	$\mathbb{Z} * \mathbb{Z}_2 * \mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	0
	$xyx^{-1}y^{-1}, x^2$	$\mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}_2)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$xyx^{-1}y^{-1}, z^2$	$\mathbb{Z}^2 * \mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
	$xyx^{-1}y, z^2$	$(\mathbb{Z} \ltimes \mathbb{Z}) * \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}_2$	0
	x^3, y^3	$\mathbb{Z} * \mathbb{Z}_3^{*2}$	$\mathbb{Z} \oplus \mathbb{Z}_3^2$	0	$\mathbb{Z}_3^2$	0
	x^2, y^2, z^2	$\mathbb{Z}_2^{*3}$	$\mathbb{Z}_2^3$	0	$\mathbb{Z}_2^3$	0
x, y	x^7	$\mathbb{Z} * \mathbb{Z}_7$	$\mathbb{Z} \oplus \mathbb{Z}_7$	0	$\mathbb{Z}_7$	0
	one relator, length 7, not proper power			0	0	0
	x^5, y^2	$\mathbb{Z}_2 * \mathbb{Z}_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	0
	$x^2yx^{-1}y, y^2$	S_3	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$ or $\mathbb{Z}_2$	0
	x^4, y^3	$\mathbb{Z}_3 * \mathbb{Z}_4$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4$	0	$\mathbb{Z}_3 \oplus \mathbb{Z}_4$	0
	$xyx^{-1}y^{-1}, x^3$	$\mathbb{Z} \times \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
	$xyx^{-1}y, y^3$	$\mathbb{Z}_3 \ltimes \mathbb{Z}$	$\mathbb{Z}$	0	0 or $\mathbb{Z}_3$	0 or $\mathbb{Z}_3$
	x^3y, y^3	$\mathbb{Z}_9$	$\mathbb{Z}_9$	0	$\mathbb{Z}_9$	0
x	x^8	$\mathbb{Z}_8$	$\mathbb{Z}_8$	0	$\mathbb{Z}_8$	0

Table 6: The groups of length 9 with 1, 2 or 3 generators

(ii) if $q \equiv 1, 2 \pmod 4$,

$$H_q(G) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We have $H_1(G) \cong \langle x, y | xyx^{-1}y, y^n \rangle_{ab} \cong \langle x, y | y^2, y^n \rangle_{ab} \cong \mathbb{Z} \oplus \langle y | y^2, y^n \rangle$, and $\langle y | y^2, y^n \rangle \cong \mathbb{Z}_2$ if n is even, 0 if n is odd.

Suppose that $q \geq 2$, and let us compute $H_q(G) = H_q(G, \mathbb{Z})$. Let $N = \{y^k : k \in \mathbb{Z}_n\} \cong \mathbb{Z}_n$ and $M = \{x^j : j \in \mathbb{Z}\} \cong \mathbb{Z}$. The sets N and M are subgroups of G, and N is a normal subgroup of G, but M is not a normal subgroup of G if $n \geq 3$. The group G is the semidirect product $N \ltimes M$ of groups N and M. We have $G = \{x^j y^k : j \in \mathbb{Z}, k \in \mathbb{Z}_n\}$, with the product given by $(x^j y^k)(x^i y^m) = x^{i+j} y^{(-1)^i k + m}$ if $j, i \in \mathbb{Z}$, $k, m \in \mathbb{Z}_n$. In particular, if $j \in \mathbb{Z}$ and $k \in \mathbb{Z}_n$, we have $x^{-j} y^k x^j = y^{(-1)^j k}$.

We have an exact sequence of groups

$$0 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 0$$

with $G/N = \{x^j N : j \in \mathbb{Z}\} \cong \mathbb{Z}$. By the Lyndon-Hochschild-Serre theorem (see e.g. [27, p. 661]), there is a first quadrant spectral sequence with

$$E^2_{p,r} = H_p(G/N, H_r(N, \mathbb{Z})) \Longrightarrow H_{p+r}(G, \mathbb{Z}).$$

Since $G/N \cong \mathbb{Z}$ is free, we have, for all modules A and all $p \geq 2$, $H_p(G/N, A) \cong 0$, so, for all $p \geq 2$, $E^2_{p,r} \cong 0$. Then it is known [27, p. 642] that there is an exact sequence

$$0 \longrightarrow E^2_{0,q} \longrightarrow H_q(G, \mathbb{Z}) \longrightarrow E^2_{1,q-1} \longrightarrow 0$$

We have $H_q(N, \mathbb{Z}) \cong H_q(\mathbb{Z}_n, \mathbb{Z}) \cong 0$ if q is even, $q \geq 2$, so $E^2_{0,q} \cong 0$ if q is even, $q \geq 2$, and $E^2_{1,q-1} \cong 0$ if q is odd, $q \geq 3$. Thus

$$H_q(G, \mathbb{Z}) \cong \begin{cases} E^2_{0,q} = H_0(G/N, H_q(N, \mathbb{Z})) & \text{if } q \text{ is odd, } q \geq 3 \\ E^2_{1,q-1} = H_1(G/N, H_{q-1}(N, \mathbb{Z})) & \text{if } q \text{ is even, } q \geq 2 \end{cases}$$

where $H_q(N, \mathbb{Z}) \cong \mathbb{Z}_n$ is a G/N-module for the action induced on $H_q(N, \mathbb{Z})$ by conjugation. The action by conjugation of $x^j N \in G/N$ on $y^k \in N$ is given by $(x^j N) \cdot y^k = x^{-j} y^k x^j = y^{(-1)^j k}$. In this case, it is known that the action induced by conjugation by x is multiplication by $(-1)^i$ on $H_{2i-1}(N, \mathbb{Z})$ [31, p. 191]. Thus, if $q \equiv 3 \pmod 4$, G/N acts trivially on $H_q(N, \mathbb{Z})$, and if $q \equiv 1 \pmod 4$, G/N acts non-trivially. Since $G/N \cong \mathbb{Z}$, we have $\mathbb{Z}(G/N) \cong \mathbb{Z}[x, x^{-1}]$, and we will write additively $H_q(N, \mathbb{Z}) \cong \mathbb{Z}_n$. Then the non-trivial action of $P(x) \in \mathbb{Z}[x, x^{-1}]$ on $k \in \mathbb{Z}_n$ is given by $P(x) \cdot k = P(-1)k$ (Note that, for $n = 2$, $G \cong \mathbb{Z} \times \mathbb{Z}_2$ and G/N acts trivially on $\mathbb{Z}_2$).

For the trivial action of G/N on $\mathbb{Z}_n$, we have $H_0(G/N, \mathbb{Z}_n) \cong H_1(G/N, \mathbb{Z}_n) \cong \mathbb{Z}_n$, so $H_q(G, \mathbb{Z}) \cong \mathbb{Z}_n$ if $q \equiv 0, 3 \pmod 4$.

For the non-trivial action of G/N on $\mathbb{Z}_n$, we have, by lemmas A and B below,

$$H_0(G/N, \mathbb{Z}_n) \cong H_1(G/N, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Lemma A. *Let G/N act non-trivially on $\mathbb{Z}_n$, by $(x^l N) \cdot k = (-1)^l k$. Then*

$$H_0(G/N, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Proof. It is known that, for any group K and any $\mathbb{Z}K$-module A, we have $H_0(K, A) \cong A/\Delta A$, where Δ is the fundamental ideal of $\mathbb{Z}K$.

Here, $H_0(G/N, \mathbb{Z}_n) \cong \mathbb{Z}_n/\Delta\mathbb{Z}_n$, where $\Delta = (x-1)\mathbb{Z}[x, x^{-1}]$. Then $(x-1)P(x) \in (x-1)\mathbb{Z}[x, x^{-1}]$ acts on $k \in \mathbb{Z}_n$ by $(x-1)P(x) \cdot k = (-1-1)P(-1)k = -2P(-1)k$, so $\Delta\mathbb{Z}_n = \{-2k : k \in \mathbb{Z}_n\}$. We have

$$\{-2k : k \in \mathbb{Z}_n\} \cong \begin{cases} \mathbb{Z}_{n/2} & \text{if } n \text{ is even} \\ \mathbb{Z}_n & \text{if } n \text{ is odd} \end{cases}$$

and so we have

$$\mathbb{Z}_n/\Delta\mathbb{Z}_n \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Lemma B. *Let G/N act non-trivially on $\mathbb{Z}_n$, by $(x^l N) \cdot k = (-1)^l k$. Then*

$$H_1(G/N, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Proof. It is known that, for any group K and any $\mathbb{Z}K$-module A, we have $H_1(K, A) \cong \ker\beta$, where $\beta : A \otimes_K \Delta \to A$ is defined by $\beta(a \otimes_K (g-1)) = (g-1)a$, if $a \in A$, $g \in K$ [12].

Here $H_1(G/N, \mathbb{Z}_n) \cong \ker \beta$, where $\beta : \mathbb{Z}_n \otimes_{G/N} \Delta \to \mathbb{Z}_n$ is defined by

$$\beta(k \otimes_{G/N} (x-1)P(x)) = (x-1)P(x) \cdot k = -2P(-1)k,$$

if $k \in \mathbb{Z}_n$, $(x-1)P(x) \in \Delta$. There is an isomorphism $\psi : \mathbb{Z}_n \otimes_{G/N} \Delta \to \mathbb{Z}_n$, defined by

$$\psi(k \otimes_{G/N} (x-1)P(x)) = P(-1)k,$$

with inverse given by $\psi^{-1}(k) = k \otimes_{G/N} (x-1)$. This isomorphism maps $\ker \beta$ onto $\ker(\beta \circ \psi^{-1})$, and $\beta \circ \psi^{-1}(k) = \beta(k \otimes_{G/N} (x-1)) = -2k$, so

$$\ker(\beta \circ \psi^{-1}) = \{k \in \mathbb{Z}_n : -2k = 0\} = \begin{cases} \{0, n/2\} & \text{if } n \text{ is even} \\ \{0\} & \text{if } n \text{ is odd} \end{cases}$$

and we are done. $\square$

7 Lower bounds for $b_q(N)$

Groups $\mathbb{Z}^n$ have presentations $\langle x_1, \ldots, x_n | ([x_i, x_j])_{1 \le i < j \le n} \rangle$ of length $n + 4\binom{n}{2} = 2n^2 - n$, and $H_q(\mathbb{Z}^n) \cong \mathbb{Z}^{\binom{n}{q}}$, so

$$b_q(\mathbb{Z}^n) = n(n-1)\cdots(n-q+1)/q!$$

Thus we have the following proposition.

Proposition 7.1 *For all $q \ge 2$, all $n \ge 1$,*

$$b_q(2n^2 - n) \ge n(n-1)\cdots(n-q+1)/q!$$

In particular, $b_3(15) \ge 1$, $b_4(28) \ge 1$.

We will see that it is interesting to get numbers n_q such that $b_q(N) > N$ if $N \ge n_q$. The following proposition achieves this task.

Proposition 7.2 *For all $q \ge 3$, let m_q be defined by*

$$n \ge m_q \iff n(n-1)\cdots(n-q+1)/q! \ge 2(n+1)^2 - (n+1),$$

and let $n_q = 2m_q^2 - m_q$. Then, for all $N \ge n_q$, $b_q(N) > N$.

Proof. Let $N \geq n_q$, and let n be such that $2n^2 - n \leq N < 2(n+1)^2 - (n+1)$. Then $n \geq m_q$, and $b_q(N) \geq b_q(2n^2 - n) \geq b_q(\mathbb{Z}^n) = n(n-1)\cdots(n-q+1)/q! \geq 2(n+1)^2 - (n+1) > N$. $\square$

Corollary 7.3 *We have*

(i) $b_3(N) > N$ *if* $N \geq 561$,

(ii) $b_4(N) > N$ *if* $N \geq 231$,

(iii) $b_9(N) > N$ *if* $N \geq 325$.

The lower bounds given above are not impressive, but better lower bounds would require ingenious arguments. A referee suggested following the original proof of Nabutovsky and Weinberger [25]. We begin with an abelian group G with N independent generators and, by many effective embeddings, we get the double suspension S^2G such that $H_3(S^2G) \cong H_1(G) \cong G$, so $b_3(S^2G) \geq N$. By keeping track of the lengths of the presentations of the groups involved in this construction, we can get an upper bound length$(S^2G) \leq n$, so we have $b_3(n) \geq b_3(S^2G) \geq N$.

8 Prospects and conclusion

8.1 Computation of $b_q(10)$

The list of groups of length 10 with more than two generators can be made and does not provide groups with $b_q(G) > 0$ if $q \geq 3$. But the groups of length 10 with two generators are numerous and the value of $b_q(10)$ for $q \geq 3$ is an open problem.

8.2 Looking for groups G with large $b_q(G)$

Only a thorough exploration of groups of length $\leq N$ can lead to the computation of $b_q(N)$. But guessing groups G with high $b_q(G)$ gives lower bounds for $b_q(N)$. The search for such groups can be done with pencil and paper or by computer. The practice of busy beaver competition has shown that computer is more effective. For the value $\Sigma(6)$ of the busy beaver function, for instance, pencil and paper searches gave $\Sigma(6) \geq 42$, while computer searches gave $\Sigma(6) \geq 4.6 \times 10^{1439}$ [23].

There exists a software package that computes homology of groups. The HAP package for the GAP system [9, 11] can compute the homology of many finite groups and certain infinite groups. Unfortunately, it is not designed to directly compute homology groups of infinite groups given by generators and relations, which are the groups that are interesting for the present problem.

8.3 Still higher: the c function

For a group G, Nabutovsky and Weinberger [25] defined $j(G)$ as the number of homology groups of G with an infinite rank.

$$j(G) = \operatorname{card}(\{q \in \mathbb{N} : b_q(G) = \infty\}).$$

Then they defined function $c(N)$ as the maximum $j(G)$ among finitely presented groups G of length $\leq N$ and finite $j(G)$.

$$c(N) = \max(\{j(G) : G \text{ finitely presented of length } \leq N, \text{ and } j(G) < \infty\}).$$

They proved that function c grows as the fifth busy beaver function B_5. As we saw in Section 4, this makes sense even if no formal definition is given for function B_5. The results from Section 6 show that, for all groups of length ≤ 9 and all $q \in \mathbb{N}$ we have $b_q(G) < \infty$, so we have the following proposition.

Proposition 8.1 *If* $0 \leq N \leq 9$*, then* $c(N) = 0$*.*

Stallings [30] gave a finitely presented group G of length ≤ 39, such that $b_3(G) = \infty$, and Bieri [4] stated that $b_q(G) < \infty$ if $q \neq 3$, so $j(G) = 1$. We can deduce the following lower bound: $c(39) \geq 1$.

8.4 The biggest number ever written

Functions b_q and c enable us to give candidates for the biggest number ever written with a limited number of symbols, say eight symbols. Write $f^k(n)$ for the iterate $f(f(\cdots f(n)\cdots))$ k times. If f grows rapidly, and if $f(n) > n$ for all $n \geq n_0$, then $k \mapsto f^k(n_0)$ grows much more rapidly than f. So it is valuable to look for numbers n_q such that $b_q(N) > N$ for all $N \geq n_q$. We saw in Section 7 that $n_9 = 325$ is suitable. So $b_9^{9^9}(9^9)$ is certainly a very big number, possibly the biggest one that has ever been written with eight symbols.

We have currently no idea for the number m such that $c(N) > N$ for all $N \geq m$. Are $c^{9^9}(9^{9^9})$ or $c^{9^{9^9}}(9^9)$ very big? And which is the biggest one? We can suspect, as a referee does, that the latter one is much bigger than the former one, and that both these numbers are much bigger than $b_9^{9^9}(9^9)$, but the proofs are still to be found.

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